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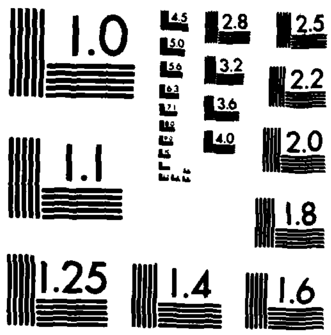

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ITEM #20, CONTINUED:

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A DECOMPOSITION OF THE BETA DISTRIBUTION, RELATED ORDER AND ASYMPTOTIC BEHAVIOR

by

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Abstract

Let $\underset{\sim}{\beta}_{v,w}$ be any beta variate with p.d.f. $\frac{\Gamma(v+w)}{\Gamma(v)\Gamma(w)} x^{v-1}(1-x)^{w-1}$ and let $\underset{\sim}{U}_{v,w} = -\log \underset{\sim}{\beta}_{v,w}$. Then $\underset{\sim}{U}_{v,w} = \underset{\sim}{U}^{CM} + \underset{\sim}{U}^{PF}$, where $\underset{\sim}{U}^{CM}$ and $\underset{\sim}{U}^{PF}$ are independent with completely monotone and PF_{∞} densities, respectively. It is shown that $\underset{\sim}{U}_{v,w}$ is infinitely divisible and $\underset{\sim}{\beta}_{v,w}$ correspondingly infinitely factorable. The asymptotoc behavior of $\underset{\sim}{U}_{v,w}$ and $\underset{\sim}{\beta}_{v,w}$ for large v, w is described. For different modes of increase of v and w, $\underset{\sim}{U}_{v,w}$ is asymptotically normal, gamma or extreme value distributed. The decomposition is employed to provide an algorithm for generating random $\underset{\sim}{\beta}_{v,w}$ distributed numbers. Many of the results are based on insights provided by the classical theory of the Gamma function in the complex plane.

KEY WORDS: Beta variates, infinite divisibility, asymptotic convergence, limit theorems, random number generation

§0. Introduction and summary

The Beta distribution plays a key role in multivariate analysis [2], ~~[12]~~ and in order statistics [14]. A useful tool for the asymptotic study of the beta variate $\underset{\sim}{\beta}_{v,w}$ is its logarithm $\underset{\sim}{U}_{v,w} = -\log \underset{\sim}{\beta}_{v,w}$ which, as we will see, has simple structural properties. The beta variate $\underset{\sim}{\beta}_{v,w}$ has p.d.f.

$$(0.1) \qquad f_{\underset{\sim}{\beta};v,w}(x) = \frac{1}{B(v,w)} x^{v-1}(1-x)^{w-1} , \quad 0 < x < 1 , \quad 0 < v,w ,$$

where $B(v,w)$ is the beta function $B(v,w) = \Gamma(v)\Gamma(w)/\Gamma(v+w)$. Correspondingly, the density for $\underset{\sim}{U}_{v,w}$ is

$$(0.2) \qquad f_{\underset{\sim}{U};v,w}(x) = e^{-x} f_{\underset{\sim}{\beta};v,w}(e^{-x}) = \frac{e^{-vx}(1 - e^{-x})^{w-1}}{B(v,w)} ,$$

$$0 < x < \infty , \quad 0 < v,w .$$

The variate $\underset{\sim}{U}_{v,w}$ has the simple generating function $\phi_{\underset{\sim}{U};v,w}(s) = E[e^{-s\underset{\sim}{U}}] = E[\underset{\sim}{\beta}^s] = \int_0^1 f_{\underset{\sim}{\beta}}(x)x^s dx$, i.e.,

$$(0.3) \qquad \phi_{\underset{\sim}{U};v,w}(s) = \frac{\Gamma(v+w)}{\Gamma(v)} \frac{\Gamma(v+s)}{\Gamma(v+w+s)} , \quad \mathrm{Re}(s) > -v .$$

The transform (0.3) is the basis for the asymptotic and structural study which follows. We will see that $\underset{\sim}{U}$ and hence $\underset{\sim}{\beta}$ have a simple decomposition. One finds that

$$(0.4) \qquad \underset{\sim}{U} = \underset{\sim}{U}^{CM} + \underset{\sim}{U}^{PF} ,$$

where $\underset{\sim}{U}^{CM}$ and $\underset{\sim}{U}^{PF}$ are independent. When $\underset{\sim}{U}^{CM} \neq \underset{\sim}{0}$, it has a completely

monotone p.d.f. When $\underset{\sim}{U}^{PF} \neq \underset{\sim}{0}$, it has a p.d.f. which is PF_∞ in the notation of total positivity [4]. On the basis of this decomposition, one sees that $\underset{\sim}{U}$ is infinitely divisible, and $\underset{\sim}{\beta}$ infinitely factorable in the corresponding sense.

The powerful apparatus of the Gamma function in the complex plane permits one to find the asymptotic behavior of $\underset{\sim}{U}_{v,w}$ and $\underset{\sim}{\beta}_{v,w}$ as v and w go to infinity. The behavior is simple and interesting. It will be shown that

(a) $\underset{\sim}{U}_{v_0,w} - \log w \xrightarrow{d} \underset{\sim}{G}$ as $w \to +\infty$ for $v_0 > 0$,

(b) $v\underset{\sim}{U}_{v,w_0} \xrightarrow{d} \underset{\sim}{\chi}_{w_0}$ as $v \to +\infty$ for $w_0 > 0$,

(c) $\underset{\sim}{Z}_{v,w} = \dfrac{\underset{\sim}{U}_{v,w} - \mu_{\underset{\sim}{U};v,w}}{\sigma_{\underset{\sim}{U};v,w}} \xrightarrow{d} \underset{\sim}{N}(0,1)$ as $v,w \to +\infty$ for a broad simple family of paths given in Section 2.

In (a), $\underset{\sim}{G}$ is a conjugate transform of an extreme value variate, $\underset{\sim}{\chi}_w$ is the gamma variate of parameter w, and $\underset{\sim}{N}(0,1)$ is the standard normal variate.

The explicit numerical evaluation of the distribution of a product of independent betas arising in multivariate analysis under the normality assumption [2], [12] can be expedited with the help of the corresponding variate $\underset{\sim}{U}_j = -\log \underset{\sim}{\beta}_j$, whose sums map into the desired product. The independent sum needed is a multifold convolution which can be performed with speed and accuracy by the Laguerre transform method [7], [8].

§1. A basic decomposition of beta variates and associated infinite factorability

The principal objective of this section is the following theorem.

Theorem 1.1

$\underset{\sim}{U} = -\log \underset{\sim}{\beta}$ is infinitely divisible for any $\underset{\sim}{\beta}$ variate. Equivalently, any $\underset{\sim}{\beta}$ is infinitely factorable, i.e., $\underset{\sim}{\beta} = \underset{\sim}{\delta}_{n1} \cdot \underset{\sim}{\delta}_{n2} \cdots \underset{\sim}{\delta}_{nn}$ where $\underset{\sim}{\delta}_{nj}$ are i.i.d.

We will prove this theorem through a lemma which provides insight into the structure of beta variates and is of some interest in its own right.

Lemma 1.2

Let $w = [w] + \theta$, where $[a]$ is the largest integer less than or equal to a and $0 \le \theta \le 1$. Then one has the decomposition

$$(1.1) \qquad \underset{\sim}{U}_{v,w} = \underset{\sim}{U}^{CM}_{v,\theta} + \underset{\sim}{U}^{PF}_{v+\theta,[w]} \quad ,$$

where (a) $\underset{\sim}{U}^{CM}_{v,\theta}$ and $\underset{\sim}{U}^{PF}_{v+\theta,[w]}$ are independent; (b) $\underset{\sim}{U}^{CM}_{v,\theta}$ has a completely monotone p.d.f. when $0 < \theta < 1$ and $\underset{\sim}{U}^{CM}_{v,0} = \underset{\sim}{0}$. Furthermore, $\underset{\sim}{U}^{CM}_{v,\theta} \to \frac{1}{v}\underset{\sim}{E}$ as $\theta \to 1$, where $\underset{\sim}{E}$ is an exponential variate with mean one; (c) $\underset{\sim}{U}^{PF}_{v,\theta,[w]}$ has a PF_∞ p.d.f. when $[w] = 1,2,3\ldots$ and $\underset{\sim}{U}^{PF}_{v+\theta,0} = \underset{\sim}{0}$.

Proof

From (0.3), one has

$$(1.2) \qquad \phi_{\underset{\sim}{U}}(s) = \left\{\frac{\Gamma(v+\theta)}{\Gamma(v)}\,\frac{\Gamma(v+s)}{\Gamma(v+\theta+s)}\right\}\left\{\frac{\Gamma(v+\theta+[w])}{\Gamma(v+\theta)}\,\frac{\Gamma(v+\theta+s)}{\Gamma(v+\theta+[w]+s)}\right\}$$

$$= \phi_{\underset{\sim}{U};v,\theta}(s) \cdot \phi_{\underset{\sim}{U};v+\theta,[w]}(s) \quad ,$$

i.e., $\underset{\sim}{U}_{v,w} = \underset{\sim}{U}_{v,\theta} + \underset{\sim}{U}_{v+\theta,[w]}$ and $\underset{\sim}{\beta}_{v,w} = \underset{\sim}{\beta}_{v,\theta} \cdot \underset{\sim}{\beta}_{v+\theta,[w]}$, where $\underset{\sim}{U}_{v,\theta}$ and $\underset{\sim}{U}_{v+\theta,[w]}$ are independent and $\underset{\sim}{\beta}_{v,\theta}$ and $\underset{\sim}{\beta}_{v+\theta,[w]}$ are independent. The density of $\underset{\sim}{U}_{v,\theta} = -\log \underset{\sim}{\beta}_{v,\theta}$ is, from (0.2), $f_{\underset{\sim}{U};v,\theta}(y) = e^{-vy}(1 - e^{-y})^{\theta-1}$. Consequently,

$$(1.3) \qquad f_{\underset{\sim}{U};v,\theta}(y) = \frac{1}{B(v,\theta)} \sum_{k=0}^{\infty} p_{\theta k} e^{-(k+v)y} \quad ,$$

where $p_{\theta 0} = 1$ and $p_{\theta k} = \prod_{j=1}^{K} [1 - (\theta/j)]$, $0 \le \theta < 1$, $k \ge 1$, so that $f_{\underset{\sim}{U};v,\theta}(y)$ is completely monotone. We write $\underset{\sim}{U}_{v,\theta} = \underset{\sim}{U}^{CM}_{v,\theta}$. We note from (1.2) that, for Re $s > -v$, $\phi_{U;v,\theta}(s) \to 1$ as $\theta \to 0$ and therefore $\underset{\sim}{U}^{CM}_{v,0} = \underset{\sim}{0}$. Similarly, $\phi_{U;v,\theta}(s) \to \frac{v}{s+v}$ as $\theta \to 1$ and $\underset{\sim}{U}^{CM}_{v,\theta} \overset{d}{\to} \frac{1}{v} \underset{\sim}{E}$ as $\theta \to 1$, proving (a) and (b). For (c), we see that, for $[w] \ge 1$, $\phi_{\underset{\sim}{U};v+\theta,[w]}(s) = \prod_{j=0}^{[w]-1} (v+\theta+j)/(s+v+\theta+j)$. For the variate $\underset{\sim}{U}_{v+\theta,[w]} = -\log \underset{\sim}{\beta}_{v+\theta,[w]}$, one has therefore

$$(1.4) \qquad \underset{\sim}{U}_{v+\theta,[w]} = \sum_{j=0}^{[w]-1} \frac{1}{v+\theta+j} \underset{\sim}{E}_j$$

where the $\underset{\sim}{E}_j$ are independent exponential variates with $E[\underset{\sim}{E}_j] = 1$. It follows that $\underset{\sim}{U}_{v+\theta,[w]}$ has PF density [4], when $[w] \ge 1$, and we write $\underset{\sim}{U}^{PF}_{v+\theta,[w]}$. From (1.2) we see that $\underset{\sim}{U}^{PF}_{v+\theta,0} = \underset{\sim}{0}$, proving the lemma. □

<u>Proof of Theorem 1.1</u>

It has been shown by F. Steutel [13] that any completely monotone variate is infinitely divisible. Since $\underset{\sim}{E}$ is infinitely divisible and the sum of infinitely divisible variates is infinitely divisible, the result is immediate. □

The decomposition of Lemma 1.2 shows that $f_{\underset{\sim}{U}}(x)$ is the convolution of a strongly unimodal p.d.f. [3], the PF component, and a completely monotone component, shedding additional light on the familiar unimodality of all beta variates.

Remark 1.3

As shown in [9], any p.d.f. $f_{\underset{\sim}{X}}(x)$ with the decomposition (1.1) has the property that $f_{\underset{\sim}{X}}(x)*f_{-\underset{\sim}{X}}(x)$, where the asterisk denotes convolution, is a scale mixture of symmetric normals, and that for such a distribution, distance to normality is measured by the kurtosis of $\underset{\sim}{X}$. The kurtosis of $\underset{\sim}{U}$, therefore, provides a consistent measure of the log-normality of $\underset{\sim}{\beta}$ described in the next section.

The decomposition (1.1) has also been demonstrated in [6] for any passage time $\underset{\sim}{T}_{mn}$ between any two states m, n of any birth-death process.

§2. Asymptotic behavior of beta variates for large v and w

We turn next to the asymptotic behavior of the $\underset{\sim}{U}$ and $\underset{\sim}{B}$ variates.

Theorem 2.1

Let $w = [w] + \theta$, $0 \le \theta < 1$. Then for any $v > 0$, $\underset{\sim}{U}_{v,w} - \log w \xrightarrow{d} \underset{\sim}{G}$ as $w \to +\infty$ where the p.d.f. of $\underset{\sim}{G}$ is given by $f_{\underset{\sim}{G}}(y) = (e^{-(v-1)y}/\Gamma(v))\cdot e^{-y}e^{-e^{-y}}$, $-\infty < y < \infty$.

Proof

Let $K = [w] - 1$ so that $w = K+1+\theta$. Let $\underset{\sim}{S}_{v,w} = \underset{\sim}{U}^{PF}_{v+\theta,[w]} - \log w$. Then, from (1.4), the Laplace transform of the p.d.f. of $\underset{\sim}{S}_{v,w}$ is given by $\phi_{\underset{\sim}{S};v,w}(s) = \prod_{j=0}^{K} (v+\theta+j)(K+1+\theta)^s/(s+v+\theta+j)$. This can be rewritten as

$$(2.1)\qquad \phi_{\underset{\sim}{S};v,w}(s) = \frac{v+\theta}{s+v+\theta}\left\{\prod_{j=1}^{K} \frac{j}{s+v+\theta+j} K^{s+v+\theta}\right\}\cdot \left\{\prod_{j=1}^{K} \frac{v+\theta+j}{j} K^{-(v+\theta)}\right\}\cdot\left(1+\frac{1+\theta}{K}\right)^s .$$

The first bracket in (2.1) converges to $\Gamma(1+s+v+\theta)$ while the second converges to $1/\Gamma(1+v+\theta)$ as $K \to +\infty$. Hence, for $0 \le \theta < 1$ fixed, one has

$$(2.2)\qquad \phi_{\underset{\sim}{S};v,w}(s) \longrightarrow \frac{\Gamma(s+v+\theta)}{\Gamma(v+\theta)} \quad \text{as} \quad w \to +\infty .$$

From (1.2) one has $\phi_{\underset{\sim}{U};v,\theta}(s) = \frac{\Gamma(v+\theta)}{\Gamma(v)}\cdot\frac{\Gamma(v+s)}{\Gamma(v+\theta+s)}$. Since $\underset{\sim}{U}_{v,w} - \log w = \underset{\sim}{U}^{CM}_{v,\theta} + \underset{\sim}{U}^{PF}_{v+\theta,[w]} - \log w = \underset{\sim}{U}^{CM}_{v,\theta} + \underset{\sim}{S}_{v,w}$, one has, from (2.2) for every fixed θ $(0 \le \theta < 1)$,

$$(2.3)\qquad \phi_{\underset{\sim}{U};v,w}(s) = \phi_{\underset{\sim}{U};v,\theta}(s)\cdot\phi_{\underset{\sim}{S};v,w}(s) \to \frac{\Gamma(v+s)}{\Gamma(v)}, \text{ as } w \to +\infty ,$$

where this limit is independent of θ. □

We note that the extreme value distribution, whose p.d.f. is $f_{\underset{\sim}{L}}(y) = e^{-y}, e^{-e^{-y}}$, $-\infty < y < \infty$, has the bilateral Laplace transform $\phi_{\underset{\sim}{L}}(s) = \Gamma(1+s)$. Hence the limit of both $\phi_{\underset{\sim}{S};v,w}(s)$ in (2.2) and $\phi_{\underset{\sim}{U};v,w}(s)$ in (2.3) are conjugate transforms [5] of $\phi_{\underset{\sim}{L}}(s)$.

Theorem 2.1 above describes the asymptotic convergence in distribution of $\underset{\sim}{U}_{v,w}$ - log w for v fixed as $w \to +\infty$, to an extreme value variate. In the next theorem, we deal with the asymptotic behavior for w fixed as $v \to +\infty$, and show convergence in distribution of $v\underset{\sim}{U}_{v,w}$ to a Gamma variate. Finally, in Theorem 2.5 we will be dealing with sequences (v_n, w_n) in which both v_n and w_n become infinite in a specified way, and asymptotic normality will be demonstrated. The three cases are shown graphically in Fig. 1 (a), (b), (c).

<u>Theorem 2.2</u>

Let $w = [w] + \theta > 0$ be fixed where $0 \le \theta < 1$. Then one has $v\underset{\sim}{U}_{v,w} \overset{d}{\to} \chi_w$, as $v \to +\infty$, where χ_w is the gamma variate with p.d.f. $f_{\chi,w}(s) = x^{w-1}e^{-x}/\Gamma(w)$.

<u>Proof</u>

Let $f^{CM}_{\underset{\sim}{U};v,\theta}(y)$ be the p.d.f. of $\underset{\sim}{U}^{CM}_{v,\theta}$. Then, from (0.2), $f^{CM}_{\underset{\sim}{U};v,\theta}(y) = \frac{1}{B(v,\theta)} e^{-vy}(1 - e^{-y})^{-(1-\theta)}$. The p.d.f. of $v\underset{\sim}{U}^{CM}_{v,\theta}$ is then given by

$$\frac{1}{v} f^{CM}_{\underset{\sim}{U};v,\theta}\left(\frac{y}{v}\right) = \frac{1}{v} \frac{\Gamma(v+\theta)}{\Gamma(v)\Gamma(\theta)} \cdot \frac{e^{-y}}{(1 - e^{-y/v})^{1-\theta}}$$

$$= \frac{1}{\Gamma(\theta)} y^{\theta-1} e^{-y} \cdot \frac{\Gamma(v+\theta)}{v^{\theta}\Gamma(v)} \cdot \frac{1}{\left(\frac{1 - e^{-y/v}}{y/v}\right)^{1-\theta}} \quad .$$

From the Stirling formula, $\Gamma(v) \sim \sqrt{2\pi}\, v^{v-\frac{1}{2}} e^{-v}$ as $v \to +\infty$, one has

$$\frac{1}{v} f^{CM}_{\underset{\sim}{U};v,\theta}\left(\frac{y}{v}\right) \to \frac{1}{\Gamma(\theta)} y^{\theta-1} e^{-y} \quad \text{as} \quad v \to +\infty \quad ,$$

i.e., $v\underset{\sim}{U}^{CM}_{v,\theta} \xrightarrow{d} \chi_\theta$ as $v \to \infty$. For the PF part $\underset{\sim}{U}^{PF}_{v+\theta,[w]}$ with $[w] \geq 1$, one has from (1.4) that

$$v\underset{\sim}{U}^{PF}_{v+\theta,[w]} = \sum_{j=1}^{[w]-1} \frac{1}{1 + \frac{\theta+j}{v}} \underset{\sim}{E}_j \xrightarrow{d} \chi_{[w]} \quad .$$

The theorem now follows from Lemma 1.2. □

From (0.3), one has

$$\text{(2.4a)} \qquad \mu_{\underset{\sim}{U};v,w} = E[\underset{\sim}{U}_{v,w}] = -\frac{d}{ds} \log \phi_{\underset{\sim}{U};v,w}(s)\Big|_{s=0} = \psi(v+w) - \psi(v) \quad ,$$

$$\text{(2.4b)} \qquad \sigma^2_{\underset{\sim}{U};v,w} = \mathrm{Var}[\underset{\sim}{U}_{v,w}] = \left(\frac{d}{ds}\right)^2 \log \phi_{\underset{\sim}{U};v,w}(s)\Big|_{s=0} = \psi'(v) - \psi'(v+w) \quad ,$$

where

$$\text{(2.5)} \qquad \psi(z) = \frac{d}{dz} \log \Gamma(z) = \Gamma'(z)/\Gamma(z) \quad , \quad \mathrm{Re}\ z > 0 \quad .$$

Let

$$\text{(2.6)} \qquad \underset{\sim}{Z}_{v,w} = (\underset{\sim}{U}_{v,w} - \mu_{\underset{\sim}{U};v,w})/\sigma_{\underset{\sim}{U};v,w} \quad .$$

We next show that $\underset{\sim}{Z}_{v,w} \to N(0,1)$ as v and w go to $+\infty$ along certain paths. Two preliminary lemmas are needed.

<u>Lemma 2.3</u>

$$\frac{x}{1 - e^{-x}} < 1+x \quad \text{for all} \quad x > 0 \quad .$$

Proof

It is clear that $1+x < \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$ for all $x > 0$. Then $(1+x)e^{-x} < 1$ so that $x < (1+x)(1 - e^{-x})$, and the result follows. □

Lemma 2.4

Let P be any directed path in the (v,w) plane for which either (a) $v,w \to +\infty$ and $v \le Kw^a$, $0 < a < 1$, $K > 0$, or (b) $v,w \to +\infty$ and $v \ge Kw^a$, $a \ge 1$, $K > 0$. Then $v\sigma_{\underset{\sim}{U};v,w} \to +\infty$ as $v,w \to +\infty$ along P.

Proof

We note the following identity [1]

$$(2.7) \qquad \psi'(z) = \int_0^\infty \frac{te^{-zt}}{1 - e^{-t}} dt , \quad \mathrm{Re}\, z > 0 \quad .$$

Then, from (2.4b), $v^2\sigma^2_{\underset{\sim}{U};v,w} = v^2 \int_0^\infty \frac{t}{1 - e^{-t}} e^{-vt}(1 - e^{-wt})dt$. Hence, since $\frac{1 - e^{-t}}{t} < 1$ for $t > 0$, we have

$$v^2\sigma^2_{\underset{\sim}{U};v,w} \ge v \int_0^\infty e^{-x}(1 - e^{-\frac{w}{v}x})dx \quad .$$

If $v \le Kw^a$, $0 < a < 1$, $K > 0$, one has $w/v \ge w^{1-a}/K$ and by the dominated convergence theorem $\int_0^\infty e^{-x}(1 - e^{-wx/v})dx \to 1$ along such a path. For the case (b), one sees that $w/v \le w^{1-a}/K$, $a \ge 1$, $K > 0$ and $w/v \to 0$ along P. One then has

$$v^2\sigma^2_{\underset{\sim}{U};v,w} \ge K^2 w \int_0^\infty e^{-x} \frac{1 - e^{-\frac{w}{v}x}}{w/v} dx \quad .$$

The integral in the last term converges to $\int_0^\infty xe^{-x}dx = 1$ by the dominated convergence theorem and the lemma follows. □

We are now ready to show asymptotic normality of $\underset{\sim}{U}_{v,w}$ under the conditions of Lemma 2.4.

Theorem 2.5

$\underset{\sim}{Z}_{v,w} \to \underset{\sim}{N}(0,1)$ as v and w go to $+\infty$ along any path P as given in Lemma 2.4.

Proof

We write $\mu = \mu_{\underset{\sim}{U};v,w}$ and $\sigma = \sigma_{\underset{\sim}{U};v,w}$ for notational simplicity. It is clear from (2.6) that

$$(2.8) \qquad \phi_{\underset{\sim}{Z};v,w}(s) = E[e^{-s\underset{\sim}{Z}}] = e^{\frac{s}{\sigma}\mu}\phi_{\underset{\sim}{U};v,w}(\tfrac{s}{\sigma}) \quad .$$

We note that for sufficiently small $|s|$, one has from (0.3) and (2.5) $\phi_{\underset{\sim}{U};v,w}(x) = \exp[\int_{v}^{v+w} \psi(u)du - \int_{v+s}^{v+w+s} \psi(u)du] = \exp[\int_{v}^{v+s} \psi(u)du - \int_{v+w}^{v+w+s} \psi(u)du]$ so that

$$(2.9) \qquad \phi_{\underset{\sim}{U};v,w}(s) = \exp[\int_0^s \{\psi(v+x) - \psi(v+w+x)\}dx] \quad .$$

$\phi_{Z;v,w}(s)$ in (2.8) can then be rewritten from (2.9) as $\phi_{\underset{\sim}{Z};v,w}(s) = \exp[\int_0^{s/\sigma} \{(\psi(v+x) - \psi(v)) - (\psi(v+w+x) - \psi(v+w))\}dx]$. By letting $y = \sigma x$, we obtain

$$(2.10) \qquad \psi_{\underset{\sim}{Z};v,w}(s) = e^{\int_0^s h(v,w,y)dy} \quad .$$

where

$$(2.11) \qquad h(v,w,y) = \frac{1}{\sigma}\int_0^{y/\sigma} \{\psi'(v+u) - \psi'(v+w+u)\}du \quad .$$

It will be seen that $\frac{d}{dy} h(v,w,y) \to 1$ for all $y > 0$ as $v,w \to +\infty$ along the path given and that one then has $\phi_{\underset{\sim}{Z};v,w}(s) \to e^{\frac{1}{2}s^2}$, as needed.

From (2.11), $\frac{d}{dy} h(v,w,y) = \frac{1}{\sigma^2}\{\psi'(v + y/\sigma) - \psi'(v + w + y/\sigma)\}$ so that (2.4b) and (2.7) lead to

$$(2.12) \qquad \frac{d}{dy} h(v,w,y) = \frac{\int_0^\infty \frac{t}{1 - e^{-t}} e^{-(v + y/\sigma)t}(1 - e^{-wt})dt}{\int_0^\infty \frac{t}{1 - e^{-t}} e^{-vt}(1 - e^{-wt})dt} .$$

We note that $\frac{d}{dy} h(v,w,y)$ is monotone decreasing in y $(y > 0)$, and $0 \le \frac{d}{dy} h(v,w,y) \le 1$ for all $v,w,y > 0$. Let $x = vt$. Then (2.12) becomes

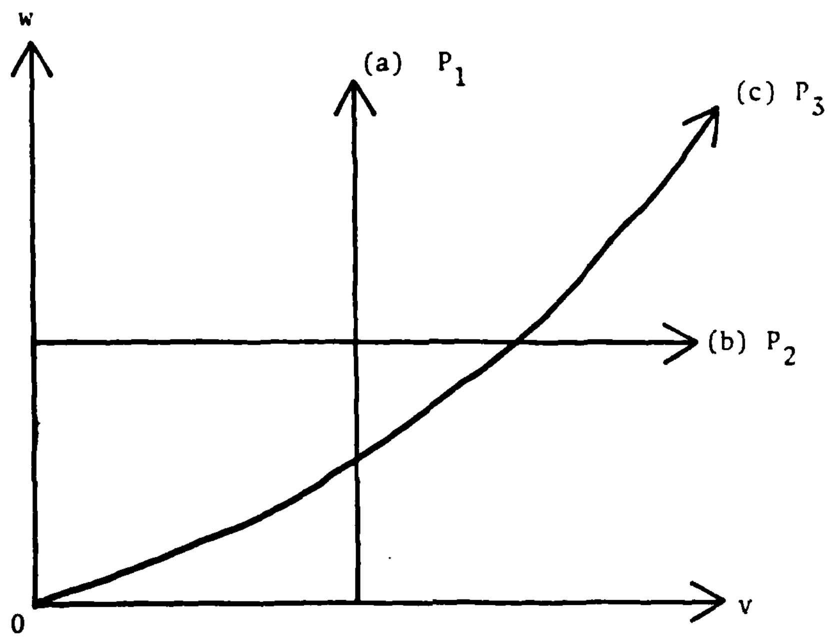
$$(2.13) \qquad \frac{d}{dy} h(v,w,y) = \frac{\int_0^\infty \frac{x/v}{1 - e^{-x/v}} e^{-(1 + y/\sigma_v)x}(1 - e^{-\frac{w}{v}x})dx}{\int_0^\infty \frac{x/v}{1 - e^{-x/v}} e^{-x}(1 - e^{-\frac{w}{v}x})dx} .$$

From Lemma 2.3, $\frac{x/v}{1 - e^{-x/v}} < 1 + \frac{x}{v}$ for $x,v > 0$ and by the dominated convergence theorem one can pass v to the limit along the path given. It follows from Lemma 2.4 that $\frac{d}{dy} h(v,w,y) \to 1$ as $v,w \to +\infty$ along the path given. From (2.11), $h(v,w,0) = 0$ so that $h(v,w,y) = \int_0^y \frac{d}{du} h(v,w,u)du$. Since $0 \le \frac{d}{dy} h(v,w,y) \le 1$, one sees that $h(v,w,y) \to y$ as $v,w \to +\infty$ along the path, again by the denominated convergence theorem, for any $y > 0$. The theorem then follows. □

The convergence $v\underset{\sim}{U}_{v,w}$ described in Theorem 2.2 has been shown by G. S. Mudholkar and M. C. Trivedi (private communication). They also

state that $\underset{\sim}{U}_{v,w}$ is "asymptotically normal as $v,w \to +\infty$" but do not provide a proof [10].

In the original form of Theorem 2.5, only ray paths $v = Kw$, $K > 0$, were considered. A referee suggested the more general paths of Lemma 2.4, and indicated that the result might also be obtained from Chap. 4, Theorem 18 of V. V. Petrov [11].



(a) $\underset{\sim}{U}_{v,w} - \log w \xrightarrow{d} \underset{\sim}{G}$ as $w \to +\infty$ along P_1.

(b) $v\underset{\sim}{U}_{v,w} \xrightarrow{d} \underset{\sim}{\Upsilon}_w$ as $v \to +\infty$ along P_2.

(c) $$\underset{\sim}{Z}_{v,w} = \frac{\underset{\sim}{U}_{v,w} - \mu_{\underset{\sim}{U};v,w}}{\sigma_{\underset{\sim}{U};v,w}} \xrightarrow{d} \underset{\sim}{N}(0,1)$$

as $v,w \to +\infty$ along P_3, where P_3 is a path such that either $v \le Kw^a$, $0 < a < 1$, $K > 0$ or $v \ge Kw^a$, $a \ge 1$, $K > 0$.

Fig. 1. Asymptotic Behavior of $\underset{\sim}{U}_{v,w} = -\log \underset{\sim}{\beta}_{v,w}$

§3. Generation of $\underset{\sim}{\beta}_{v,w}$ random numbers

The decomposition $\underset{\sim}{U}_{v,w} = \underset{\sim}{U}^{CM}_{v,\theta} + \underset{\sim}{U}^{PF}_{v+\theta,[w]}$ in Lemma 1.2 may be employed to provide a simple algorithm for generating $\underset{\sim}{\beta}_{v,w}$ random numbers. From (1.3) the Laplace transform of the p.d.f. of $\underset{\sim}{U}^{CM}_{v,\theta}$ can be given by

$$(3.1) \qquad \phi_{\underset{\sim}{U};v,\theta}(s) = \sum_{k=0}^{\infty} q_k \frac{v+k}{s+v+k}$$

where

$$(3.2) \qquad q_k = \frac{P_{\theta k}}{B(v,\theta)(v+k)} ; \quad P_{\theta k} = \binom{\theta-1}{k}(-1)^k , \quad k = 0,1,2,\ldots .$$

It is clear that $q_k > 0$ for all k. One sees quickly that $\sum_{k=0}^{\infty} \frac{P_{\theta k}}{v+k} = \sum_{k=0}^{\infty} \binom{\theta-1}{k} \cdot \frac{(-1)^k}{v+k} = \int_0^1 u^{v-1}(1-u)^{\theta-1} du$, i.e.,

$$(3.3) \qquad \sum_{k=0}^{\infty} \frac{P_{\theta k}}{v+1} = B(v,\theta)$$

and therefore $(q_k)_0^{\infty}$ is a probability distribution. Let $\underset{\sim}{E}_j$ be i.i.d. with the common c.d.f. $1 - e^{-x}$, $j = 0,1,\ldots,M = [w]$. From Lemma 1.2, (1.4) and (3.1), one then has

$$(3.4) \qquad \underset{\sim}{U}_{v,w} = \sum_{j=0}^{M-1} \frac{1}{v+\theta+j} \underset{\sim}{E}_j + \frac{1}{v+\underset{\sim}{N}} \underset{\sim}{E}_M$$

where $\underset{\sim}{N}$ is the discrete random variable with $P[\underset{\sim}{N} = k] = q_k$ and independent of $\underset{\sim}{E}_M$. Let $\underset{\sim}{\mathcal{U}}_j$ be independent and identical uniform variates on (0,1). Since $\underset{\sim}{\mathcal{U}}_j \overset{d}{=} e^{-\underset{\sim}{E}_j}$ and $\underset{\sim}{U}_{v,w} = -\log \underset{\sim}{\beta}_{v,w}$, Eq. (3.4) leads to

(3.5) $$\underset{\sim}{\beta}_{v,w} = \prod_{j=0}^{M-1} \underset{\sim}{U}_j^{\frac{1}{v+\theta+j}} \cdot \underset{\sim}{U}_M^{\frac{1}{\underset{\sim}{N}}} \quad .$$

Hence one has the following algorithm for generating $\underset{\sim}{\beta}_{v,w}$ random numbers.

Algorithm

(a) Generate [w]+1 independent and identical uniform variates $\underset{\sim}{U}_j(\omega)$, $j = 0,1,\ldots,M = [w]$, on $(0,1)$.

(b) Generate the variate $\underset{\sim}{N}(\omega)$ from the distribution $(q_k)_0^\infty$.

(c) $$\underset{\sim}{\beta}_{v,w}(\omega) = \prod_{j=0}^{M-1} \underset{\sim}{U}_j(\omega)^{\frac{1}{v+\theta+j}} \cdot \underset{\sim}{U}_M(\omega)^{\frac{1}{\underset{\sim}{N}(\omega)}} \quad .$$

The algorithm is simple and straightforward. Advantages and disadvantages of the algorithm with respect to existing algorithms will be described elsewhere.

§4. Explicit calculation of the distribution of the product of independent Beta variates

For certain likelihood ratio statistics arising in multivariate analysis, one must evaluate the distribution of

$$(4.1) \qquad \underset{\sim}{X} = \underset{\sim}{\beta}_{v_1,w_1} \cdot \underset{\sim}{\beta}_{v_2,w_2} \cdot \;\cdots\; \underset{\sim}{\beta}_{v_K,w_K} \;,$$

where the beta variates are independent. This distribution may be obtained via the Laguerre transform procedure described in [7], [8] in the following way. From (4.1)

$$(4.2) \qquad -\log X = \sum_{j=1}^{K} (-\log \underset{\sim}{\beta}_{v_j,w_j}) = \sum_{j=1}^{K} \underset{\sim}{U}_{v_j,w_j} \;.$$

The $\underset{\sim}{U}_{v_j,w_j}$ variates are independent and absolutely continuous with p.d.f.'s as in (0.2). They therefore have the properties of regularity and rapid decrease required by the Laguerre transform method for convolving p.d.f.'s and permit vector representations of modest length with high accuracy. The Laguerre transform coefficients required are easily obtained analytically and the calculation of the p.d.f. of $-\log \underset{\sim}{X}$ and hence of $\underset{\sim}{X}$ proceeds rapidly.

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